

A steady solution for Prandtl's self-similar vortex sheet spirals

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ABSTRACT

Prandtl's [L. Prandtl, Über die Entstehung von Wirbeln in der idealen Flüssigkeit, mit Anwendung auf die Tragflügeltheorie und andere Aufgaben, in: von Kármán, Levi-Cevita (Eds.), Vorträge aus dem Gebiete der Hydro- und Aerodynamik, Springer, Berlin, 1922] self-similar, semi-infinite, free vortex sheets are characterized by a two-dimensional unsteady flow around an invariable exponential spiral. A similar but steady spiral flow has been published by Schmidt and Sparenberg [G.H. Schmidt, J.A. Sparenberg, On the edge singularity of an actuator disk with large constant normal load, J. Ship Research 21 (1977) 125]. This flow is shown to be, kinematically, the steady solution in Prandtl's class of spirals but with a different dynamic boundary condition since it is not a free vortex sheet but carries a constant load. Due to the kinematic similarity, the analysis of Prandtl's spirals by van Kuik [G.A.M. van Kuik, The flow induced by Prandtl's self-similar vortex sheet spirals at infinite distances from the spiral kernel, Eur. J. Mech. B/Fluids 23 (4) (2004) 607–616] is also valid for the spiral with steady flow. As for Prandtl's spirals, the steady spiral flow requires a description on a multi-branched Riemann surface, by which an interpretation in the single-branched two-dimensional surface is impossible.

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1. Introduction

Prandtl [1] has published the potential of a class of two-dimensional, semi-infinite, rolled-up vortex sheets, based on the following requirements: 1-the shape of the spiraling vortex sheet has to be invariant in time (but may move), and 2-the spiral has to be force free. Expressed in polar coordinates (r, θ) the shape of the spiral is typically $r = e^{\text{constant} \cdot \theta}$. Analytical solutions for spirals that do not move have been published by Prandtl himself and by Alexander [2]. In van Kuik [3] Prandtl's spirals are discussed in detail, and a survey of the literature on these spirals is presented. The complex potential χ of Prandtl's spirals with a fixed position is given by:

$$\chi(t, z) = ct^{2m-1} z^{\alpha+i\beta}, \quad (1)$$

where $z = \frac{r}{a} e^{i\theta}$ is the dimensionless complex coordinate with a as reference length, t the time, m , α and β scalar constants. The dimension of the constant c depends on m , as will be discussed below. The analytical solutions of Prandtl and Alexander are characterized by $m = 0$.

The present paper continues the analysis in van Kuik [3], but now for the steady flow around an exponential spiral. Prandtl excludes a steady flow around such a spiral, since he shows that this spiral cannot be force-free. Schmidt and Sparenberg [4] have examined an exponential spiral with steady flow that indeed is not force-free but carries a constant load. They were looking for the

strength and shape of the vortex sheet emanating from the edge of an actuator strip with a constant load. They showed that this vortex sheet is identical to a vortex sheet with a constant load, so the potential flow problem to be solved has become: what is the shape of an infinitely thin, semi-infinite vortex sheet with a constant load? Their solution is:

$$\chi(z) = cz^{(1-i)/2}. \quad (2)$$

The shape of the spiral is $\frac{r}{a} = e^{\theta}$. It is clear that, kinematically, their solution is part of Prandtl's class of spirals since it is described by (1) with $m = \alpha = -\beta = \frac{1}{2}$. This similarity was not mentioned by Schmidt and Sparenberg themselves.

This note discusses the kinematic similarity and dynamic dissimilarity of the steady and unsteady solutions.

2. (Dis)similarity between the unsteady and steady flow spirals

2.1. Dimension analysis

Prandtl [1] has performed a dimension analysis of (1) for the analytical solution he found with $m = 0$. This is repeated here, now including the flow of Schmidt and Sparenberg with $m = \frac{1}{2}$. The dimensions of χ , t and z are $[m^2/s]$, $[s]$, and $[-]$ respectively. When submitted in (1), c appears to have a different dimension for the unsteady and steady flows: for unsteady flows $[m^2]$, for steady flows $[m^2/s]$.

Since z is defined as $z = \frac{r}{a} e^{i\theta}$, the reference length a defines the scale of the flow. The spiral itself is semi-infinite, and offers no characteristic length to define a . In this respect it resembles

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the potential flow around a semi-infinite flat plate, defined by $\chi(z) = icz^2$, $z = \frac{r}{a}e^{i\theta}$ for the plate at the positive x -axis. The scale factor a can have any value without changing the flow pattern, although the magnitude of the velocities scale with a^{-1} . To show that similar rules hold for the spiral flow, (1) is rewritten as:

$$\chi(t, z) = t^{2m-1} c^* z_{a=1}^{\alpha+i\beta} \quad (3)$$

$$\text{with } c^* = \frac{c}{a^{\alpha+i\beta}} = ca^{-\alpha} e^{-i\beta \ln a}. \quad (4)$$

The scale factor a is now included in the constant c^* , which is completely separated from the variables t and $z_{a=1}$. The potential $z_{a=1}^{\alpha+i\beta}$ determines the flow pattern which is invariant in time and does not depend on a and c . Varying the reference length a results in a rotation of the flow pattern as a whole about an angle $-\beta \ln a$, while all velocity vectors scale with $a^{-\alpha}$. Apart from the rotation of the pattern, this is similar to the semi-infinite flat plate flow: zooming the flow field in or out with an arbitrary value gives the same flow pattern, with the magnitude of the velocities depending on the zoom or scale factor. The conclusion is that the spiral flow has no characteristic length scale and, as for the semi-infinite plate, that the scale factor can be chosen arbitrarily.

A second way to scale all velocity vectors while the flow pattern rotates but otherwise remains invariant, is to change the polar angle θ . The potential (1) is non-periodic in θ as is shown by substitution of $\theta^* = \theta + 2n\pi$ (n integer) in (1):

$$z^* = \frac{r}{a} e^{i(\theta+2n\pi)}, \quad (5)$$

$$\chi(t, z^*) = e^{2n\pi(i\alpha-\beta)} \chi(t, z). \quad (6)$$

The result is the same potential multiplied by the complex constant $e^{2n\pi(i\alpha-\beta)}$. This implies a rotation of the flow pattern about an angle $2n\pi\alpha$ as well as a scaling of the velocities with a factor $e^{-2n\pi\beta}$. The potential is single-valued for only one full turn of θ , which is different from the semi-infinite flat plate flow where a 2π increase of θ does not change the potential. The spiral flow potential becomes single-valued for all turns (branches) of θ when these branches are coupled by a Riemann surface: a two-dimensional but multi-branched surface like an Archimedes- or corkscrew for which each 2π increase of θ results in a position at a new branch of the spiraling surface. In Section 3 we will analyze this.

For convenience, it will be assumed that, as in van Kuik [3], from here on $a = 1$ [m] and $n = 1$, so $c = c^*$, $z = z_{a=1}$.

2.2. The boundary conditions

In van Kuik [3] the kinematic and dynamic boundary conditions are derived for the unsteady flows. This is repeated here for spirals with a fixed position, now including the steady flow solution.

The complex potential (1) is written as

$$\begin{aligned} \Phi(t, z) + i\Psi(t, z) &= ct^{2m-1} [re^{i\theta}]^{\alpha+i\beta} \\ &= ct^{2m-1} e^{(\alpha \ln r - \beta\theta)} e^{i(\beta \ln r + \alpha\theta)} \end{aligned} \quad (7)$$

where Φ is the real potential of the flow and Ψ the stream function. The shape of the spiral is determined by $\Psi_{\text{spiral}} = 0$ which is satisfied when $\beta \ln r_{\text{spiral}} + \alpha\theta_{\text{spiral}} = 0$, giving:

$$r_{\text{spiral}} = e^{-\frac{\alpha}{\beta}\theta_{\text{spiral}}} \quad \text{for all } m. \quad (8)$$

The two sides of the spiral are denoted by $+$ and $-$, with $\theta_+ = \theta_{\text{spiral}}$ so:

$$\left. \begin{aligned} r_+ &= r_- = r_{\text{spiral}} = e^{-\frac{\alpha}{\beta}\theta_{\text{spiral}}} \\ \theta_+ &= \theta_{\text{spiral}} \\ \theta_- &= \theta_{\text{spiral}} + 2\pi \end{aligned} \right\}. \quad (9)$$

Since $\Psi_+ = \Psi_{\text{spiral}} = 0$, Ψ_- must also equal 0 for the sheet to be a stream surface. By (7) this is satisfied when $\beta \ln r_- + \alpha\theta_- = 0$ which gives, after substitution of (9):

$$\sin(2\pi\alpha) = 0. \quad (10)$$

Apparently the kinematic condition only determines α .

The dynamic boundary condition is:

$$\Delta \left(\frac{\partial \Phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} |\mathbf{v}|^2 \right) = 0 \quad (11)$$

where Δ denotes the difference between the two sides of the spiral, p is the static pressure and ρ is the flow density. In Prandtl's unsteady flows the sheet is free and cannot carry a pressure jump Δp . Therefore the pressure term in the left-hand side vanishes. In the steady flow the term $\Delta(\partial \Phi / \partial t)$ is absent while Δp may be non-zero as discussed in Section 1. The dynamic boundary condition becomes:

$$\Delta \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\mathbf{v}|^2 \right) = 0 \quad \text{for } m = 0, \quad (12)$$

$$\Delta \left(\frac{p}{\rho} + \frac{1}{2} |\mathbf{v}|^2 \right) = 0 \quad \text{for } m = \frac{1}{2}. \quad (13)$$

With χ given by (7):

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= c(2m-1)t^{2m-2} \operatorname{Re}(z^{\alpha+i\beta}) \\ &= c(2m-1)t^{2m-2} r^{\alpha} e^{-\beta\theta} \cos(\beta \ln r + \alpha\theta). \end{aligned} \quad (14)$$

The kinematic pressure $\frac{1}{2} |\mathbf{v}|^2$ equals:

$$\frac{1}{2} |\mathbf{v}|^2 = \frac{1}{2} \frac{\partial \chi}{\partial z} \overline{\frac{\partial \chi}{\partial z}} = \frac{1}{2} c^2 t^{2(2m-1)} (\alpha^2 + \beta^2) r^{2(\alpha-1)} e^{-2\beta\theta}. \quad (15)$$

Substitution of (9) in (14), (15) and combining these with (12) and (13) yields:

$$\begin{aligned} \frac{c}{2} (\alpha^2 + \beta^2) (1 - e^{-4\beta\pi}) e^{-\frac{(\alpha^2 + \beta^2 - 2\alpha)}{\beta} \theta_{\text{spiral}}} &= 1 - e^{-2\pi\beta} \cos(2\pi\alpha) \\ \text{for } m = 0, \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{c^2}{2} (\alpha^2 + \beta^2) (1 - e^{-4\beta\pi}) e^{-\frac{2(\alpha^2 + \beta^2 - \alpha)}{\beta} \theta_{\text{spiral}}} &= -\frac{\Delta p}{\rho} \\ \text{for } m = \frac{1}{2}. \end{aligned} \quad (17)$$

Eqs. (16) and (17) determine the dynamic boundary condition, and are independent of the time. Any set of constants α , β , c that makes the equations independent of θ_{spiral} constitutes a valid solution. This happens when α and β satisfy the following conditions:

$$\alpha^2 + \beta^2 - 2\alpha = 0 \quad \text{for } m = 0, \quad (18)$$

$$\alpha^2 + \beta^2 - \alpha = 0 \quad \text{for } m = \frac{1}{2}. \quad (19)$$

Together with the kinematic condition (10), the admissible values of α and β can be calculated for both values of m . The constant c in (1) is then determined by (16) or (17).

Table 1 presents the parameters of all possible solutions of (1) for spirals with fixed position. Included in the table is the inclination angle δ defined as the angle between the normal at the spiral and the radius vector from the origin: $\delta = \arctan(-\alpha/\beta)$. The spiral with $\delta = 45^\circ$ has two possible flows: unsteady with zero load or steady with constant load. All of these solutions have been previously published, but the kinematic similarity between the unsteady and steady solution has not been demonstrated before.

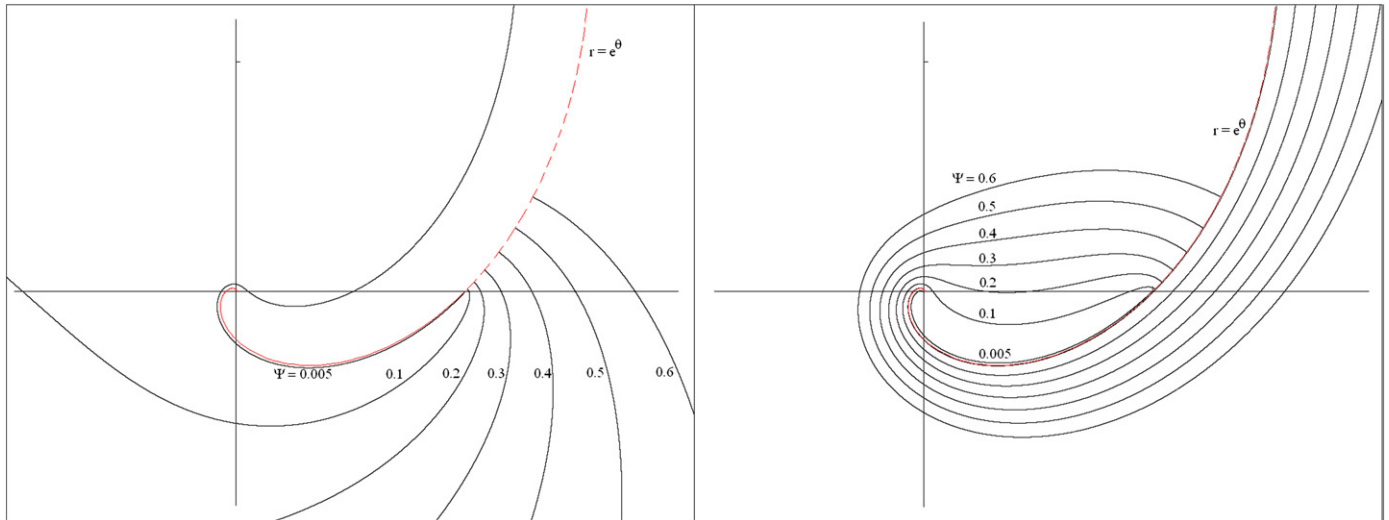


Fig. 1. The flow around the spiral $r = e^\theta$ with finite length. The centre of the spiral is the origin, the edge is given by $r = 1$; $\theta_{\text{spiral}} = 0$. The branch of the Riemann surface defined by $-2\pi < \theta \leq 0$ is shown left, and the one by $0 < \theta \leq 2\pi$ right.

Table 1
The constants defining the solutions of (1)

	m	α	β	δ	c
Prandtl [1]	0	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	30°	$2 \frac{1+e^{\pi\sqrt{3}}}{1-e^{2\pi\sqrt{3}}}$
Alexander [2]	0	1	-1	45°	$\frac{1-e^{2\pi}}{1-e^{4\pi}}$
Alexander [2]	0	$1\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	60°	$\frac{2}{3} \frac{1+e^{\pi\sqrt{3}}}{1-e^{2\pi\sqrt{3}}}$
Schmidt & Sparenberg [4]	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	45°	$\pm 2\sqrt{\frac{\Delta p/p}{e^{2\pi}-1}}$

3. The potential flow solution for the steady spiral

Since the flow pattern does not depend on the value of m , the analysis of (1) in van Kuik [3] is valid for (2) as well, by substitution of $m = \alpha = -\beta = \frac{1}{2}$. Without repetition of this analysis, the main result is discussed, now applied to the Schmidt and Sparenberg spiral.

In order to analyze the multi-valued properties of (1) and the independence of the flow pattern with respect to scale and time, the flow around a *finite* exponential spiral with the spiral centre at the origin and the edge at $(r_{\text{edge}}, \theta_{\text{edge}})$ was derived in [3]. This solution has the length of the spiral or the edge radius r_{edge} as characteristic length, and the velocity U_0 at large distance from the spiral, $r \gg r_{\text{edge}}$, as characteristic velocity. Fig. 1 shows the flow field for $-2\pi < \theta \leq 0$ and $0 < \theta \leq 2\pi$. It is clear that the potential of the finite spiral flow is multi-valued since each increase of the polar angle with 2π returns other flow parameters. At a Riemann surface the flow is continuous, but when each 2π branch is represented in a plane 2-D surface as in Fig. 1, a flow discontinuity is present across the branch line that separates two adjacent branches. The branch line is $r = e^\theta$ which coincides with the finite spiral for $r < r_{\text{edge}}$. Across the branch line a discontinuity in tangential velocity exists for $r < r_{\text{edge}}$, and a discontinuity with crossing streamlines for $r > r_{\text{edge}}$.

By taking the limit of $r_{\text{edge}} \rightarrow \infty$ (1) was recovered. Substitution of $m = \alpha = -\beta = \frac{1}{2}$ in the equations for the velocities \mathbf{v} at both sides of the finite spiral, and taking the limit of $r_{\text{edge}} \rightarrow \infty$ also recovers the pressure jump $\Delta p = \Delta \frac{1}{2} |\mathbf{v}|^2$ and the discontinuity in tangential velocity Δv for the infinite spiral (Eqs. (47) and (49) in Schmidt and Sparenberg [4]):

$$\lim_{r_{\text{edge}} \rightarrow \infty} \frac{\Delta p}{\rho} = \frac{c^2}{4} (e^{2\pi} - 1), \quad (20)$$

$$\lim_{r_{\text{edge}} \rightarrow \infty} \Delta v = \frac{-c}{\sqrt{2}} (e^\pi + 1). \quad (21)$$

This confirms that the semi-infinite spiral flow is the result of a finite spiral flow when the spiral length becomes infinite. The added value of this procedure is that it provides knowledge of the flow field at distance $O(r_{\text{edge}})$. With $r_{\text{edge}} \rightarrow \infty$ the discontinuity with crossing streamlines moves to infinity by which, at finite distance from the origin, only the discontinuity in tangential velocity remains. This makes it logical to consider the spiral as a vortex sheet instead of a branch line. However, the flow is multi-valued and (in a 2-D representation) the continuity equation is not satisfied at infinite distance from the origin. Consequently, the semi-infinite spiral flow is also a flow at a Riemann surface, with the spiral being a branch line instead of vortex sheet.

4. Conclusions

The steady spiral flow, published by Schmidt and Sparenberg [4], appears to be, kinematically, a steady solution in Prandtl's class of self-similar vortex sheet spirals [1].

As for this class of Prandtl spirals, the Schmidt and Sparenberg spiral flow is defined at a Riemann surface instead of a plane 2-D surface. In a 2-D representation, the continuity equation is not satisfied at an infinite distance from the origin.

This implies that a physical interpretation in the 2-D plane is not possible, and that the spiral flow solution is not the solution of the 2-D problem posed by Schmidt and Sparenberg: the flow at the edge of an actuator disc, or the flow induced by a vortex sheet with constant load.

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